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HOMOGENEITY OF SEVERAL SYSTEMS UNDER THE LOG-LOGISTIC DISTRIBUTION USING THE GENERALIZED TYPE II CENSORED SAMPLING DESIGN

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Abstract: In this paper, we discuss the Generalized Type II censoring scheme for the Log-Logistic distribution and obtain the Maximum Likelihood Estimate of unknown parameters. The Maximum Likelihood equations are not mathematically tractable, we use the Newton Raphson iterative procedure to obtain estimate of scale parameters, their variance covariance matrix, reliability function and hazard rate for both known and unknown shape parameters. Further, Likelihood Ratio test is used for testing the homogeneity of several scale parameters of the log logistic distribution. Monte-Carlo simulation is performed to study performance of estimates of parameters. We also carried out cost of experiment for the generalized type II censoring.

Keywords: Generalized Type II censoring, Log-Logistic distribution, Maximum Likelihood Estimation, Likelihood Ratio Test, Cost Function

1. Introduction

In statistical literature, Verhulst (1838) developed Log-Logistic ("LL") distribution to model population growth. In economics, LL distribution is well-known as the Fisk distribution due to Fisk (1961) and it is a logarithm transformation of logistic distribution. The shape parameters of this distribution resemble same property as that of lognormal distribution therefore it is analogous to lognormal distribution Further, the log-logistic distribution has heavier tails therefore it is also called heavy tailed distribution; it is rightly skewed and has narrow peak. The density function of this distribution can be expressed in closed form. Thus, it is very useful for survival data with censoring. Apart from this property, it has non-monotone hazard function when $\alpha>1$ but it is monotonically decreasing and unimodal when $\alpha \leq 1$.The hazard rate increases initially and later it decreases therefore it is said to have IFR as well as DFR. Further, The Log-Logistic distribution can be suitable substitute for the Weibull distribution. The mixture of the Gompertz as well as gamma distribution with mean and variance coincides and equal to 1 is said to follow the Log-Logistic distribution.

The industrial revolution and competitive environment has increased demands manufactured products should have good quality and reliability. To fulfill these requirements, manufacturers conduct appropriate designed experiments. In reliability study, there are various instances where obtaining a complete sample is neither desirable nor achievable due to time or cost considerations. Therefore, the practitioners terminate the experiment and report the results before all items realize their failures. The most typical sampling plan in these situations is Type II censoring. There are several research papers in the statistics literature that employ the plan for various lifetime models like as normal, exponential, and Weibull. For more information, one can refers Gupta (1952), Cohen (1965), Mann *et al.* (1974), Lawless (1982), and Hossain *et al.* (1986). (2003). In industry, the problem of comparing product efficacy is important. In this case, the reliability engineer would like to make an early and efficient choice on the effectiveness of the goods under life test in terms of standard hazard rate function after placing multiple independent samples of units manufactured by various procedures. Balakrishnan and Ng (2006) study the problem of comparing two populations using stochastic ordering extensively. Sharafi *et al.* (2013) compare the hazard rates of two distributions under Type II censoring using a distribution free test. Shanubhogue and Raykundaliya (2015) discuss the inferential problem about the lifetime of homogeneity of several systems under the generalized exponential distribution based on Type II censored sampling design, and Raykundaliya (2016) discuss generalized type II censoring scheme for Frechet distribution. and further, study the reliability characteristics of distributions. The organization of paper is as follows:

In section 2, we give the probability density function (pdf), the reliability or survival function and the hazard rate of the log-logistic distribution and develop the likelihood for the Generalized Type II censored sampling design under log-logistic distribution. In section 3, we derive the expressions for maximum likelihood estimators of parameters and their asymptotic variance-covariance matrix when shape parameter of the distribution is known and when it is unknown. Section 4 discusses algorithm for generation of data from Type II censored sampling design under log-logistic distribution and provides iterative procedure for estimation of the parameters through Newton-Raphson method. Further, the tables of ML estimates and their asymptotic standard Errors, relative variance and relative standard errors, estimate of reliability and hazard rates and their mean squares error, at fixed time point which are simulated using Monte-Carlo simulation technique for both the cases of shape parameter known and unknown. In section 5, we discuss likelihood ratio test for simultaneous testing of homogeneity of scale parameters when the shape parameter is known. The cut-off points for the test statistics are obtained through Monte-Carlo simulation. We study cost function for the experiments in section 6. The concluding remarks are given section 7.

2. The Log-Logistic Distribution, Generalized Type II Censoring Scheme and Likelihood Function for the Generalized Type II Censoring Design

Suppose T represents the lifetime of an item and follows the Log-Logistic distribution (LL) if its cumulative distribution function and probability density functions given respectively as

$$F\,(t,\alpha,\beta) \;=\; \frac{1}{\left(1+\left(\frac{t}{\beta}\right)^{-\alpha}\right)}\;;\quad (t>0,\alpha>0,\beta>0) \tag{1}$$

$$f\,(t;\,\alpha,\beta) \;=\; \frac{\left(\frac{\alpha}{\beta}\right)\left(\frac{t}{\beta}\right)^{\alpha-1}}{\left(1+\left(\frac{t}{\beta}\right)^{\alpha}\right)^{2}};\quad (t>0,\alpha>0,\beta>0) \tag{2}$$

where α is shape parameter, β is scale parameter and we can denote it as LL (α, β).

The reliability function and hazard function of LL distribution are given respectively,

$$\bar{F}(t) = \frac{\left(\frac{t}{\beta}\right)^{-\alpha}}{1+\left(\frac{t}{\beta}\right)^{-\alpha}} = \frac{1}{\left(\left(\frac{t}{\beta}\right)^{\alpha}+1\right)} \tag{3}$$

$$h(t) = \frac{f(t)}{\bar{F}(t)} = \frac{\alpha}{t\left(1+\left(\frac{t}{\beta}\right)^{-\alpha}\right)} \tag{4}$$

2.1. Random Deviate Generating Function

Let U be a random variable said to follow standard uniform distribution and cumulative distribution function $F(\cdot)$ then any sample from $F^{-1}(u)$ is drawn from $F(\cdot)$ if and only if its regular inverse exists. So, the random deviate can be generated from *LL* (α, β) using

$$t \;=\; \beta(u^{-1}\;-1)^{-\frac{1}{\alpha}} \tag{5}$$

If *Z* follows *LL*(α, 1), then the corresponding moment generating function, is given by

$$M(s) = \;\alpha\int_{-\infty}^{\infty} e^{sz}\,\frac{z^{(\alpha\,-\,1)}}{(1\,-\,z^{\alpha})^{2}}\,dz$$

$$M(s) = \;\sum_{n=0}^{\infty}\frac{t^{n}}{n!}\,B\left(1-\frac{n}{\alpha},1+\frac{n}{\alpha}\right);\ \text{where B is beta function.} \tag{6}$$

Differentiating $lnM(s)$ with respect and s and evaluating at $s = 0$, we get the mean and variance of *LL* (α, 1) as

$$E(Z) \;=\; B\left(1-\frac{n}{\alpha},1+\frac{n}{\alpha}\right)\ \text{and}\ var(Z) \;=\; B\left(1-\frac{n}{\alpha},1+\frac{n}{\alpha}\right). \tag{7}$$

If Z follows LL (α, 1) and $T = \frac{1}{\beta} Z$, then T follows LL(α, β). Therefore, the mean and variance of T is given by

$$E(T) = \beta B\left(1 - \frac{n}{\alpha}, 1 + \frac{n}{\alpha}\right) \text{ and } var(T) = \beta^2 B\left(1 - \frac{n}{\alpha}, 1 + \frac{n}{\alpha}\right) \tag{8}$$

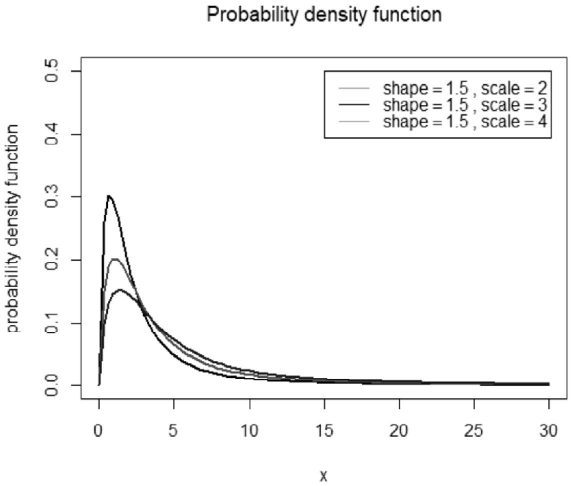

Figure 1(a): Probability Density function

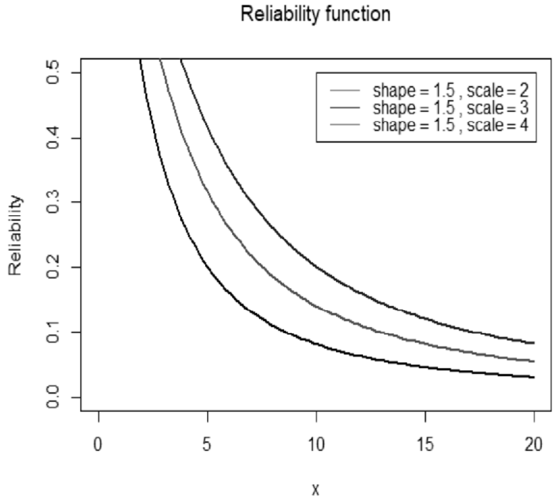

Figure 1(b): Reliability function

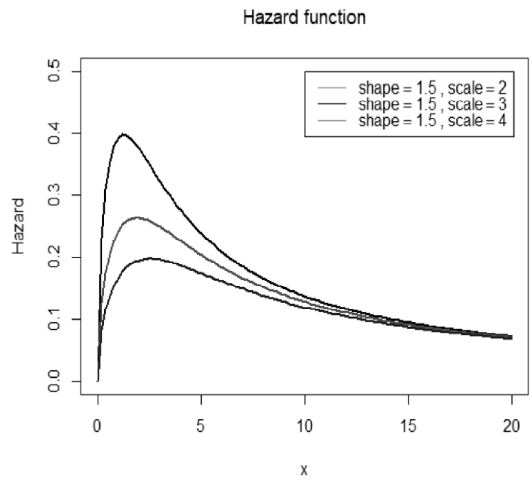

Figure 1(c): Hazard Function

Figure 1: Plots of probability density function, reliability function and hazard function of LL distribution for some specific values of parameters

2.2. Generalized Type II Censoring Scheme

We now consider a design in which we simultaneously test *m* types of systems, starting with u units for each type of system and continuing the experiment until G^* failures are observed in each type of system, i.e., the total number of units tested is mu, and the total number of failures observed at the end of the experiment is given by $G = mG^*$. Assuming that lifetime distribution of unit is Log-Logistic with shape parameter α and scale parameter β_i; $i = 1, 2,\ldots,m$ for each type of system. After each failure in the experiment, the failure time is observed and denoted as t_{gi}; $g = 1,2,\ldots, G^*$; $i = 1,2,\ldots, m$. At the end of experiments, we have data on (u, G, t_{gi}; $g = 1,2,\ldots, G^*$; $i = 1,2,\ldots, m$). Figure 2 represents the scheme of the Generalized Type II Censoring.

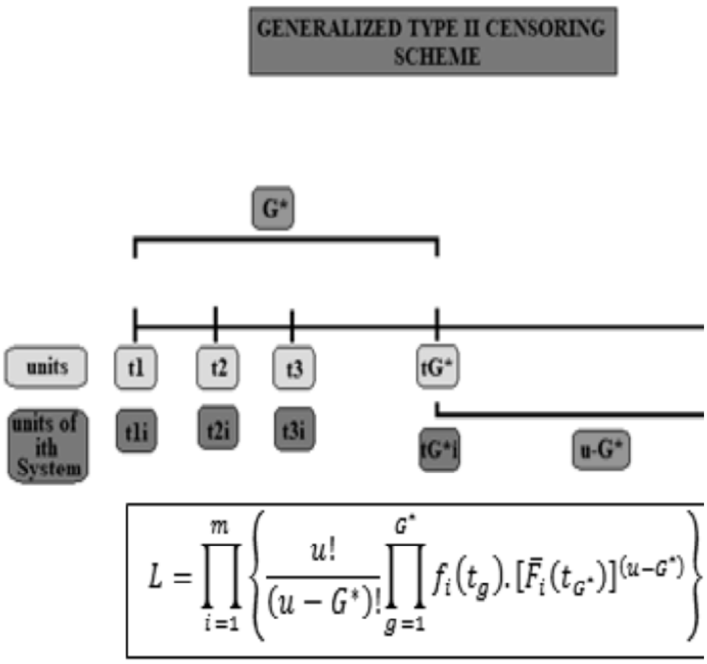

Figure 2: Graphical representation of Scheme of the Generalized Type II Censoring

2.3. Likelihood function for Generalized Type II Censoring Method

Shanubhogue and Raykundaliya (2015) defines, the likelihood function for generalized Type II censoring design for observing G^* failures from u units are as

$$\text{Li} = \frac{u!}{(u-G^*)!} \prod_{g=1}^{G^*} f_i(t_g).[\bar{F}_i(t_{G^*})]^{(u-G^*)} \tag{10}$$

Likelihood for whole experiment, as different types of systems are functioning independently, is

$$L=\prod_{i=1}^{m} L_i$$

$$L=\prod_{i=1}^{m}\left\{\frac{u!}{(u-G^*)!}\prod_{g=1}^{G^*} f_i\left(t_g\right).\left[\bar{F}_i\left(t_{G^*}\right)\right]^{(u-G^*)}\right\} \tag{11}$$

Substituting equations (1) and (2) in (11) we get the likelihood function for generalized type II censoring for LL distribution as

$$L=\prod_{i=1}^{m}\left\{\left(\frac{u!}{(u-G^*)!}\right)\prod_{g=1}^{G^*}\left(\frac{\left(\frac{\alpha}{\beta_i}\right)\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha-1}}{\left(1+\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha}\right)^2}\right)\times\left[\frac{1}{\left(\left(\frac{t_{G^*i}}{\beta_i}\right)^{\alpha}+1\right)}\right]^{(u-G^*)}\right\} \tag{12}$$

3. Maximum Likelihood Estimation

To obtain maximum likelihood estimates of, α, β_i (i = 1,2,..., m), survival function, hazard function, and observed fisher's information matrix, we use maximum likelihood estimation method. First we obtain log likelihood by taking log of likelihood function (12) which is given as

$$l=mln\left(\frac{u!}{(u-G^*)!}\right)+\sum_{i=1}^{m}\sum_{g=1}^{G^*}\left(\frac{\left(\frac{\alpha}{\beta_i}\right)\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha-1}}{\left(1+\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha}\right)^2}\right)+(u-G^*)\sum_{i=1}^{m}\left[\frac{1}{\left(\left(\frac{t_{G^*i}}{\beta_i}\right)^{\alpha}+1\right)}\right]$$

$$l=mln\left(\frac{u!}{(u-G^*)!}\right)+mG^*\ln(\alpha)-G^*\sum_{i=1}^{m}\ln(\beta_i)+(\alpha-1)\sum_{i=1}^{m}\sum_{g=1}^{G^*}\ln\left(t_{gi}\right)-G^*(\alpha-1)\sum_{i=1}^{m}\ln(\beta_i)$$

$$-2\sum_{i=1}^{m}\sum_{g=1}^{G^*}\left[\ln\left(1+\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha}\right)\right]-(u-G^*)\sum_{i=1}^{m}\left[\ln\left(\left(\frac{t_{G^*i}}{\beta_i}\right)^{\alpha}+1\right)\right] \tag{13}$$

Differentiating (13) with respect to α, β_i (i = 1, 2,..., m) we get maximum likelihood equations as

$$\frac{\partial l}{\partial\alpha}=\frac{mG^*}{\alpha}+\sum_{i=1}^{m}\sum_{g=1}^{G^*}\ln\left(\frac{t_{gi}}{\beta_i}\right)-2\sum_{i=1}^{m}\sum_{g=1}^{G^*}\left[\frac{\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha}ln\left(\frac{t_{gi}}{\beta_i}\right)}{\left(1+\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha}\right)}\right]-(u-G^*)\sum_{i=1}^{m}\left[\frac{\left(\frac{t_{G^*i}}{\beta_i}\right)^{\alpha}ln\left(\frac{t_{G^*i}}{\beta_i}\right)}{\left(1+\left(\frac{t_{G^*i}}{\beta_i}\right)^{\alpha}\right)}\right] \tag{14}$$

$$\frac{\partial l}{\partial\beta_i}=-\frac{G^*}{\beta_i}-\frac{G^*(\alpha-1)}{\beta_i}-2\alpha\sum_{g=1}^{G^*}\left[\frac{\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha-1}\left(\frac{t_{gi}}{\beta_i^2}\right)(-1)}{\left(1+\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha}\right)}\right]-(u-G^*)\alpha\left[\frac{\left(\frac{t_{G^*i}}{\beta_i}\right)^{\alpha-1}\left(\frac{t_{G^*i}}{\beta_i^2}\right)(-1)}{\left(1+\left(\frac{t_{G^*i}}{\beta_i}\right)^{\alpha}\right)}\right]$$

$$\frac{\partial l}{\partial \beta_i} = \frac{G^*}{\beta_i} + \frac{G^*(\alpha-1)}{\beta_i} - 2\alpha\sum_{g=1}^{G^*}\left[\frac{\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha-1}\left(\frac{t_{gi}}{\beta_i^2}\right)}{\left(1+\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha}\right)}\right] - (u-G^*)\alpha\left[\frac{\left(\frac{t_{G^*i}}{\beta_i}\right)^{\alpha-1}\left(\frac{t_{G^*i}}{\beta_i^2}\right)}{\left(1+\left(\frac{t_{G^*i}}{\beta_i}\right)^{\alpha}\right)}\right]$$

$$(i = 1, 2 \ldots m) \tag{15}$$

The likelihood equations (14) and (15) are not mathematically tractable, so we obtain the estimates of parameters $\underline{\beta} = (\beta_1, \beta_2,\ldots, \beta_m)$ numerically by some iterative procedure when shape parameter ? is known and unknown.

3.1. Maximum Likelihood Estimation when Shape Parameter α is known

For given values of (u, G, t_{gi}; g = 1, 2..., G^*; i = 1, 2,..., m), the solution of equations (15) can be quantitatively evaluated using an appropriate iterative process such as the Newton-Raphson technique. From these equations, the MLE, β_i is obtained numerically and denote it as $\hat{\beta}_i; i = 12, \ldots, m$. The invariance property of MLEs is used to evaluate the MLEs of reliability (R_i (t_i); i = 1, 2, ..., m) and hazard rate (h_i (t_i); i = 1, 2, ..., m) as

$$\widehat{R_i}(t_i) = \frac{1}{\left(\left(\frac{t_i}{\widehat{\beta_i}}\right)^{\alpha}+1\right)}, \tag{16}$$

Hazard function

$$\widehat{h_i}(t_i) = \frac{\alpha}{t_i\left(1+\left(\frac{t_i}{\widehat{\beta_i}}\right)^{-\alpha}\right)}. \qquad i = 1, 2, \ldots, m \tag{17}$$

3.1.1. Observed Fisher Information Matrix when Shape Parameter is known

To obtain Fisher information matrix we take derivatives of equations (15) with respect to β_i; i = 1,2,...., m. Therefore, we have

$$\frac{\partial^2 l}{\partial \beta_i^2} = \frac{mG^*}{\beta_i^2} + \frac{mG^*(\alpha-1)}{\beta_i^2} + 2\alpha\sum_{i=1}^{m}\sum_{g=1}^{G^*}\left[\frac{\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha}\left(\left(\frac{t_{gi}}{\beta_i}\right)+1+\alpha\right)}{\beta_i^2\left(1+\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha}\right)^2}\right] - (u-G^*)\alpha\sum_{i=1}^{m}\left[\frac{\left(\frac{t_{G^*i}}{\beta_i}\right)^{\alpha}\left(\left(\frac{t_{G^*i}}{\beta_i}\right)+1+\alpha\right)}{\beta_i^2\left(1+\left(\frac{t_{G^*i}}{\beta_i}\right)^{\alpha}\right)^2}\right] \tag{18}$$

$$\frac{\partial l}{\partial \beta_i \partial \beta_j} = 0.\ \forall j \neq i = 1, 2, \ldots, m. \tag{19}$$

Theorem 3.1: For given α and $\frac{G}{u}$ kept constant the maximum likelihood estimators, $\underline{\hat{\beta}}$ of $\underline{\beta}$ are consistent estimators, and $\sqrt{u}(\underline{\hat{\beta}} - \underline{\beta})$ is asymptotically m-variate normal with mean $\underline{0}$ and variance covariance matrix V^{-1},where V is expected value of negative of second derivative matrix of log likelihood with respect to β.

3.2. Maximum Likelihood Estimation When Shape Parameter ? is Unknown

Using the data (u, G, t_{gi}; g = 1, 2..., G^*; i = 1, 2, ..., m), the solution of equations (14 and 15) are obtained using the Newton-Raphson technique and obtain the MLE of (α, β_i) is derived as$(\widehat{\alpha}, \hat{\beta}_i)$, i = 12,…, m. Using invariance properties of MLEs, the MLE of reliability (R_i (t_i); i = 1, 2, ..., m) and hazard rate (h_i (t_i); i = 1, 2, ..., m) are given as

$$\widehat{R_\iota}(t_i) = \frac{1}{\left(\left(\frac{t_i}{\beta_\iota}\right)^{\widehat{\alpha}} + 1\right)} \tag{20}$$

$$\widehat{h_\iota}(t_i) = \frac{\widehat{\alpha}}{t_i\left(1 + \left(\frac{t_i}{\beta_\iota}\right)^{-\widehat{\alpha}}\right)} \tag{21}$$

3.2.1. Observed Fisher Information Matrix when Shape Parameter is unknown

To obtain Fisher information matrix we take derivatives of equations (14) and (15) with respect to α, β_i; i = 1,2,...., m. Therefore, we have,

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{mG^*}{\alpha^2} - 2\sum_{i=1}^{m}\sum_{g=1}^{G^*}\left[\frac{\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha} ln^2\left(\frac{t_{gi}}{\beta_i}\right)}{\left(1+\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha}\right)^2}\right] - (u - G^*)\sum_{i=1}^{m}\left[\frac{\left(\frac{t_{G^*i}}{\beta_i}\right)^{\alpha} ln^2\left(\frac{t_{G^*i}}{\beta_i}\right)}{\left(1+\left(\frac{t_{G^*i}}{\beta_i}\right)^{\alpha}\right)^2}\right] \tag{22}$$

$$\frac{\partial l}{\partial \alpha \partial \beta_i} = -\frac{G^*}{\beta_i} - 2\sum_{i=1}^{m}\sum_{g=1}^{G^*}\left[\frac{\left(\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha} + ln\left(\frac{t_{gi}}{\beta_i}\right)\alpha + 1\right)\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha}}{\left(1+\left(\frac{t_{gi}}{\beta_i}\right)^{\alpha}\right)^2 \beta_i}\right] - (u - G^*)\left[\frac{\left(\left(\frac{t_{G^*i}}{\beta_i^2}\right)^{\alpha} + ln\left(\frac{t_{G^*i}}{\beta_i^2}\right)\alpha + 1\right)\left(\frac{t_{G^*i}}{\beta_i^2}\right)^{\alpha}}{\left(1+\left(\frac{t_{G^*i}}{\beta_i^2}\right)^{\alpha}\right)^2 \beta_i}\right] \tag{23}$$

Derivatives of equation (15) with respect to β_i; i = 1,2,..., m and β_j; $j \neq i$ = 1,2,...,m are given in equations (18) and (19) respectively. Therefore, we have following result.

Theorem 3.2: For given α and$\frac{G}{u}$ kept constant the maximum likelihood estimators,$(\hat{\alpha}, \underline{\hat{\beta}})$ of$(\alpha, \underline{\beta}\ )$ are consistent estimators, and$\sqrt{u}(\hat{\alpha} - \alpha, \underline{\hat{\beta}} - \underline{\beta})$ is asymptotically m-variate normal with mean $\underline{0}$ and variance covariance matrix $\boldsymbol{W}^{-1}$, where $\boldsymbol{W}$ is expected value of negative of second derivative matrix of log likelihood with respect to (α, $\underline{\beta}$) .

4. Algorithm, Numerical Exploration and Conclusions

A Monte-Carlo simulation analysis is carried out in this section to compare the performance of the estimates obtained in section 3. For illustration purpose we compare $m = 2$ and $m = 3$ systems which follows failure distribution $LL(\alpha, \beta_i)$; $i = 1, 2..., m$. The R-language version R.3.1.0 was used to perform all calculations.

4.1. Known Shape Parameter

In this section, we carry out simulation study for two sets of parameter values $m = 2$, $\alpha = 1.5$, $\beta_1 = 2$, $\beta_2 =$ and for $m = 3$, $\alpha = 1.5$, $\beta_1 = 2$, $\beta_2 = 3$, $\beta_3 = 4$. The simulation is carried out for different values of u and G^*. Here we keep total number of failures in whole experiment $G = uG^*$ fixed. We simulate 1000 samples for each case. The simulation results are summarized in Table 1 and Table 2. We use following algorithm to simulate results.

4.1.1 Algorithm

Step 1: Taking $m = 2$, $\alpha = 1.5$, $\beta_1 = 2$, $\beta_2 = 3$ we generate u random numbers from LL $(\alpha, \beta_1, \beta_2, \ldots, \beta_m)$ for each types of system. The same is repeated for the parameters $m = 3$, $\alpha = 1.5$, $\beta_1 = 2$, $\beta_2 = 3$, $\beta_3 = 4$.

Step 2: Note the G^* failure times observed from each type of u copy of systems put on test and recorded as $(t_{1i}, t_{2i}, \ldots, t_{G_i^*})$; $i = 1,2..., m$ for each type of systems.

Step 3: Using data observed in step 2 and suitable initial value, evaluate $\hat{\beta}$ and corresponds sample Fisher information matrix $\hat{V}$, $S = (\frac{\partial l}{\partial \underline{\beta}})$: score vector.

Step 4: Use Newton-Raphson iterative method

$$\hat{\beta}_{New} = \hat{\beta}_{Old} - \hat{V}^{-1}(\hat{\beta}_{Old}) * S$$

Step 5: Repeat Step 5 until the $\sum_{i=1}^{m} |\hat{\beta}_{iNew} - \hat{\beta}_{iOld}| < \epsilon$ where ϵ is very small, predefined quantity.

Step 6: Repeat the procedures in Step 1 to Step 5 for 1000 times and obtain following quantities.

(a) $EV_i = \frac{\sum_{j=1}^{N} \hat{\beta}_{ij}}{N}$

(b) Mean Squared Error, $MSE_i = \frac{\sum_{j=1}^{N} (\hat{\beta}_{ij} - \beta_i)^2}{N}$ where β_i; $i = 1,2,..,m$ the values of parameters given in Step 1.

(c) Average of Variance-Covariance Matrices computed for different simulated samples, say $\hat{V}^{*-1}$.

(d) Reliability functions $\hat{R}_{ij}(t_i)$ and hazard rate $\hat{h}_{ij}(t_i)$; *i*= 1, 2..., *m*; *j* = 1, 2..., N evaluate using equations (16-17) and corresponding MSEs are

$$\frac{\sum_{j=1}^{N}\left(\hat{R}_{ij}(t_i)-R_i(t_i)\right)^2}{N} \text{ and } \frac{\sum_{j=1}^{N}\left(\hat{h}_{ij}(t_i)-h_i(t_i)\right)^2}{N}$$

(e) Relative Variance(RV) = $\frac{MSE_i}{EV_i}$, Bia s = $EV_i - \beta_i$ and Relative Standard Error (RSE) = $\frac{\sqrt{MSE_i}}{EV_i}$.

Step 7: Obtain Standard Error (SE) of estimates by taking square root of diagonal elements of V $\hat{V}^{*-1}$.

we obtained the results which are shown in Tables (4.1) and (4.2), and concluded that the average value of MLE's for scale parameters *i* = 1, 2,..., *m*, reliability characteristic, and hazard rate are close to their true values for given shape parameters. Furthermore, the average mean square Errors are minimal. We also notice that as u increases, the estimates converse to its true value and their SE, RSE, RV decreases.

4.2. Unknown Shape Parameter

Similar study, with not much change in the algorithm, one can do simulation study for the case of unknown shape parameter. We perform simulation studies for the set of parameters values $m = 2$, $\alpha = 1.5$, $\beta_1 = 2$, $\beta_2 = 3$ and $m = 3$, $\alpha = 1.5$, $\beta_1 = 2$, $\beta_2 = 3$, $\beta_3 = 4$.

From the above Table (4.3) and Table (4.4) we can easily notice that in the presence of unknown shape parameter α, the MLEs of scale parameters β_i; *i* = 1, 2, …, *m*, the reliability characteristics and hazard rates are closer to their actual values. Since, the shape parameter is unknown the convergence rate is comparatively slower than that of known shape parameter. Perhaps, it may be the effect of estimate of unknown shape parameter ?. Further, we can say, somewhat large sample size is required than what we consider for the estimates to reach their true values.

5. Testing of Hypotheses

The generalized type II censoring design is said to be significant only when can prove that all m type of systems has non-identical lifetime. This can be accomplished by applying an ANOVA approach for the suggested design. However, we will develop a test using the likelihood approach. The goal of the hypothesis testing problem is to test

H_0: $\beta_1 = \beta_2 = \ldots = \beta_m = \beta$ *vs* H_1 : $\beta_i \neq \beta_j$ for at least one pair (i, j) $i \neq j = 12,\ldots,m$ (24)

The likelihood ratio test statistic to test H_0 defines as

$$\lambda_{LR} = \frac{\max_{\alpha,\beta} L(\underline{t},\beta,\alpha)}{\max_{\alpha,\underline{\beta}} L(\underline{t},\underline{\beta},\alpha)}$$

The test based on $-2ln\ (\lambda_{LR})$ rejects H_0 in support of H_1 if it is larger than upper α-th cut-off point of chi-square distribution $(m - 1)$ degrees of freedom.

5.1. Computation of Likelihood Under H0

The log likelihood lnL_Gunder null hypothesis from equation (13) we have,

$$l = mln\left(\frac{u!}{(u-G^*)!}\right) + mG^*\ln(\alpha) - mG^*\ln(\beta) + (\alpha-1)\sum_{i=1}^{m}\sum_{g=1}^{G^*}\ln(t_{gi}) - mG^*(\alpha-1)\ln(\beta)$$

$$-2\sum_{i=1}^{m}\sum_{g=1}^{G^*}\left[\ln\left(1+\left(\frac{t_{gi}}{\beta}\right)^{\alpha}\right)\right] - (u-G^*)\sum_{i=1}^{m}\left[\ln\left(\left(\frac{t_{G^*i}}{\beta}\right)^{\alpha}+1\right)\right] \quad (25)$$

Differentiate (25) with respect to α and β we have

$$\frac{\partial l}{\partial \alpha} = \frac{mG^*}{\alpha} + \sum_{i=1}^{m}\sum_{g=1}^{G^*}\ln(t_{gi}) - mG^*ln(\beta) - 2\sum_{i=1}^{m}\sum_{g=1}^{G^*}\left[\frac{\left(\frac{t_{gi}}{\beta}\right)^{\alpha} ln\left(\frac{t_{gi}}{\beta}\right)}{\left(1+\left(\frac{t_{gi}}{\beta}\right)^{\alpha}\right)}\right] - (u-G^*)\sum_{i=1}^{m}\left[\frac{\left(\frac{t_{G^*i}}{\beta}\right)^{\alpha} ln\left(\frac{t_{G^*i}}{\beta}\right)}{\left(1+\left(\frac{t_{G^*i}}{\beta}\right)^{\alpha}\right)}\right] \quad (26)$$

$$\frac{\partial l}{\partial \beta} = \frac{mG^*}{\beta} + \frac{mG^*(\alpha-1)}{\beta} - 2\alpha\sum_{g=1}^{G^*}\left[\frac{\left(\frac{t_{gi}}{\beta}\right)^{\alpha-1}\left(\frac{t_{gi}}{\beta^2}\right)}{\left(1+\left(\frac{t_{gi}}{\beta}\right)^{\alpha}\right)}\right] - (u-G^*)\alpha\left[\frac{\left(\frac{t_{G^*i}}{\beta}\right)^{\alpha-1}\left(\frac{t_{G^*i}}{\beta^2}\right)}{\left(1+\left(\frac{t_{G^*i}}{\beta}\right)^{\alpha}\right)}\right] \quad (27)$$

Differentiate (26) and (27) with respect to (α, β) and α we have

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{mG^*}{\alpha^2} - 2\sum_{i=1}^{m}\sum_{g=1}^{G^*}\left[\frac{\left(\frac{t_{gi}}{\beta}\right)^{\alpha} ln^2\left(\frac{t_{gi}}{\beta}\right)}{\left(1+\left(\frac{t_{gi}}{\beta}\right)^{\alpha}\right)^2}\right] - (u-G^*)\sum_{i=1}^{m}\left[\frac{\left(\frac{t_{G^*i}}{\beta}\right)^{\alpha} ln^2\left(\frac{t_{G^*i}}{\beta}\right)}{\left(1+\left(\frac{t_{G^*i}}{\beta}\right)^{\alpha}\right)^2}\right] \quad (28)$$

$$\frac{\partial^2 l}{\partial \beta^2} = \frac{mG^*}{\beta^2} + \frac{mG^*(\alpha-1)}{\beta^2} + 2\alpha\sum_{i=1}^{m}\sum_{g=1}^{G^*}\left[\frac{\left(\frac{t_{gi}}{\beta}\right)^{\alpha}\left(\left(\frac{t_{gi}}{\beta}\right)+1+\alpha\right)}{\beta^2\left(1+\left(\frac{t_{gi}}{\beta}\right)^{\alpha}\right)^2}\right] - (u-G^*)\alpha\sum_{i=1}^{m}\left[\frac{\left(\frac{t_{G^*i}}{\beta}\right)^{\alpha}\left(\left(\frac{t_{G^*i}}{\beta}\right)+1+\alpha\right)}{\beta^2\left(1+\left(\frac{t_{G^*i}}{\beta}\right)^{\alpha}\right)^2}\right] \quad (29)$$

$$\frac{\partial l}{\partial \alpha \partial \beta} = -\frac{mG^*}{\beta} - 2\sum_{i=1}^{m}\sum_{g=1}^{G^*}\left[\frac{\left(\left(\frac{t_{gi}}{\beta}\right)^{\alpha} + ln\left(\frac{t_{gi}}{\beta}\right)\alpha+1\right)\left(\frac{t_{gi}}{\beta}\right)^{\alpha}}{\left(1+\left(\frac{t_{gi}}{\beta}\right)^{\alpha}\right)^2\beta}\right] - (u-G^*)\left[\frac{\left(\left(\frac{t_{G^*i}}{\beta^2}\right)^{\alpha} + ln\left(\frac{t_{G^*i}}{\beta^2}\right)\alpha+1\right)\left(\frac{t_{G^*i}}{\beta^2}\right)^{\alpha}}{\left(1+\left(\frac{t_{G^*i}}{\beta^2}\right)^{\alpha}\right)^2\beta}\right]$$

(30)

The likelihood equation (27) is not mathematically tractable for known as well as unknown shape parameter we use the Newton-Raphson method to obtain the estimate of parameter β. Here we deal with only known shape parameter. We demonstrate the test procedure for $m = 2$ and $m = 3$. We generate data under our design for the parameter values under H_1: $\alpha = 1.5$, $\beta_1 = 2$, $\beta_2 = 3$ and H_1: $\alpha = 1.5$, $\beta_1 = 2$, $\beta_2 = 3$, $\beta_3 = 4$ respectively. Then carry out the test procedure as suggested above. The procedure is repeated for the different choices of u and G^*. The results are produced in the Table 5.1 and Table 5.2, respectively.

From the Table 5.1, we infer that the power the test is poor for small sizes. Higher the sample size is required to exhibits its power in identifying the alternative. From Table 5.2, it can reveal that for comparing homogeneity of three systems, as sample size becomes 72 it exhibits its power.

6. The Cost Function under Generalized Type II Censoring Design

To study the cost of experiment on the line of Srivastava (1987), with modified notation we proceed at follows:

Let us begin testing experiment with u systems, each from m brands. Further, we shall assume that the cost of a system failing is constant, say C_1, regardless of the brand. Since, the experiment terminates after observing total $G = mG^*$ where G^* is a fixed number of failures being observed on each brand, the total cost of failure is $G = mC_1\ G^*$, which is fixed and pre-planned. There are m different sub experiments under this system, in the sense that how we observe machines of type i is independent of the failures of machines of typei', for $i \neq i'$. Let $t_{G^*i} = 1,2, \ldots, m$ denote the time of failure of G^*th machine of the ith type. Define t_{max} by $t_{max} = max\ t_{max} = \max\ (t_{G^*1}, t_{G^*2}, \ldots t_{G^*m})$,

As a result, t_{max} reflects the period that the entire experiment under Type II censoring lasts. In this experimental technique, the cost component is from the perspective of the experiment's length, is $C_3\ t_{max}$

Now, we will concentrate at the cost of total experimental time for all the units under test. Assume C_2 is the cost per unit time associated with a machine's testing time, regardless of brand. Let $t_{1i}, t_{2i}, \ldots t_{G^*i}$ be the failure times of G^* systems out of u systems of type i, $i = 1,2\ldots\ldots m$.

Then the total time the on test for the sub experiment is

$$ut_{1i} + (\mathrm{u}-1)(t_{2i} - t_{1i}) + (\mathrm{u}-2)(t_{3i} - t_{2i}) + \cdots. + (\mathrm{u} - G^* + 1)\,(t_{G^*i} - t_{(G^*-1)i})$$

Hence, the total machine time under Type II censoring is t, were

$$t_2 = \sum_{i=1}^{n} t_{1i} + t_{2i} +, \ldots, +(u - G^* + 1)t_{G^*i}$$

The contribution of this cost is $t_2\ C_2$.

The fourth cost component is the total number of machines utilized in the experiment, which equals $mu = N$ under generalized Type II censoring. This expense covers the purchase, storage, and handling of the machines. We will consider $\gamma(N)$ to be an increasing function of N. is the cost of putting N machines through their paces. In some circumstances, $\gamma(N)$ may increase at a slower rate than N or possibly be non-existent.

Finally, let us denote the experiment's overhead costs as C_0. We will presume that this C_0 represents general costs like those related to preparing the experiment, administrative and consulting costs, and so on. However, we assume that C_0 is independent of N, the total number of machines used in the experiments. Under the generalized Type II censoring design, the actual cost of the experiment under these assumptions is

$$C_a{}^{II} = C_0+C_1G+C_2t_2+C_3t_{max}+\gamma(N).$$

For the cost effectiveness of the generalized Type II censoring method, we perform Monte Carlo simulations with cost values of $C_0 = 100$, $C_1 = 5$, $C_2 = 10$, $C_3 = 10$ and $\gamma(N) = 0.5N$. The results for $m = 2$ and $m = 3$ are shown in Tables 6.1 and 6.2, respectively. We also change the number of units to be tested at each sub experiment and fix the overall number of failures G, Table 6.3 contains the results.

6.1. Algorithm for Evaluation of Total Time under Experiment, Duration of Experiment and Total Cost

Step 1: Generate the subset of Type II censored observation using Step 1 and Step 2 of the algorithm given in section 4.1.1.

Step 2: Repeat Step 1, $n = 1000$ times and obtain average failure time and store it as $(t_{1i}, t_{2i},\ldots t_{G^*i})$; i = 1, 2, …, m.

Step 3: Evaluate total time under experiment and duration of experiment using formulae by $t_2 = \sum_{i=1}^{n} t_{1i} + t_{2i} +, \ldots, +(u - G^* + 1)t_{G^*i}$ and t_{max} respectively.

Step 4: Use the values of C_0, C_1, C_2, C_3 and $\gamma(N) = 0.5N$ where $N = um$ in cost function

$C_a{}^{II} = C_0 + C_1G + C_2t_2 + C_3t_{max} + \gamma(N)$d obtain the different values of total cost for different combination of (u, G).

It can be seen from Table 6.4 and Table 6.5. total time under experiment, duration of the experiment and the cost associated with the experiment in generalized Type II censoring scheme are increasing functions of G, for $m = 2$ as well as $m = 3$. From Table 6.3, we can see that as proportion of censoring decreases the total time under experiment increases linearly and the duration time of the experiment decreases.

7. Concluding Remarks

In this paper we study estimation of parameters through iterative procedure and study their performance when life time of random variable is LL distribution under generalized type II censoring scheme. We observe that estimate of parameters converse to its true value when sample size increases in both case when shaper parameter is known as well as unknown. The performance indicators like MSE, Bias, RSE decreases with sample size and support the performance of estimators. The

Likelihood Ratio test is also exhibits its power to attain its alternative hypothesis when number of items put on test increases especially when comparing more than two types of systems. Further, form study of the cost of experiment, we observed that cost of experiment, duration of experiment increases linearly with total number of test items increases with fixed proportion of censoring. It is also observed that for fixed items censoring, cost of experiment and total time on experiment increases marginally with increases test items but duration of experiment decreases.

References

[1] Balakrishnan, N. and Ng, H.K.T. (2006). Precedence-type tests and applications, Hoboken NJ: John Wiley & Sons.

[2] Cohen, A.C. Jr. (1965). Maximum likelihood estimation with Weibull distribution based on complete and on censored samples, Technometrics, 7, 559–588.

[3] Fisk, P.R. (1961), "The Graduation of Income Distributions", *Econometrica*, **29** (2): 171–185, doi:10.2307/1909287, JSTOR 1909287

[4] Gupta, A.K. (1952). Estimation of mean and standard deviation of a normal population from a censored sample, Biometrika, 39, 260–273.

[5] Hossain, A and Zimmer, W.(2003). Comparison of estimation methods for Weibull parameters: Completed censored samples, Journal of Statistical Computation and Simulation, 73(2), 145-153.

[6] Lawless, J.F. (1982). Statistical Models and Methods for Lifetime Data, John Wiley & Sons, New York, NY.

[7] Mann, N., Schafer, E. and Singpurwalla, N.(1974). Methods for Statistical Analysis of Reliability and Life Data, John Wiley & Sons, New York, NY.

[8] Raykundaliya, D.P, (2016). Inferential Problem about Homogeneity of Several Systems under Frechet Distribution, Journal of Statistics Application and Probability.

[9] Sharafi, M.M Balkrishnan, N. And Khaledi, B.E. (2013). Distribution-free comparison of hazard rates of two distributions under Type II censoring, Communications in Statistics-Theory and Methods, 42, 1889–1898.

[10] Srivastava, J.N. (1987). More efficient and less time consuming censoring designs for life testing. J.Statist.Plann.Inference. Vol.16, pp.389-413

[11] Shanubhogue,A., Raykundaliya, D.P., (2015), A study of inferential problem about the lifetime of homogeneity of several systems under generalized exponential model based on type-II censored sampling design, ProbStat Forum, Volume 08, Pages 24–33.

[12] Verhulst, Pierre-François (1838). "Notice sur la loi que la population suit dans son accroissement". *Correspondance mathématique et physique*. **10**: 113–121.

Table 4.1: Maximum Likelihood Estimate of Parameters, Reliability and Hazard Rates and their Efficiency Measures $m = 2$, $\alpha = 1.5$, $\beta_1 = 2$, $\beta_2 = 3$, $R(t) = (0.5, 0.5)$, $h(t) = (0.375, 0.250)$

u	G^*		$\widehat{\beta_1}$	$\widehat{\beta_2}$	$\hat{R}_1(t_1)$	$\hat{R}_2(t_2)$	$\hat{h}_1(t_1)$	$\hat{h}_2(t_2)$
12	6	**EV**	2.0996	3.0186	0.5024	0.4913	0.3723	0.2564
		MSE	0.3612	0.7924	0.0047	0.0050	0.0057	0.0027
		SE	0.7560	1.0772	--	--	--	--
		RV	0.1720	0.2625	0.0094	0.0102	0.0154	0.0105
		Bias	0.0996	0.0186	0.0024	-0.0087	-0.0027	0.0064
		RSE	0.2862	0.2949	0.1367	0.1442	0.2032	0.2027
24	12	**EV**	2.1707	2.8585	0.5155	0.4827	0.3578	0.2628
		MSE	0.2147	0.3865	0.0026	0.0029	0.0033	0.0016
		SE	0.5480	0.7276	--	--	--	--
		RV	0.0989	0.1352	0.0051	0.0061	0.0091	0.0061
		Bias	0.1707	-0.1415	0.0155	-0.0173	-0.0172	0.0128
		RSE	0.2135	0.2175	0.0996	0.1122	0.1595	0.1524
36	18	**EV**	2.2115	2.8299	0.5220	0.4821	0.3505	0.2633
		MSE	0.1634	0.2497	0.0302	0.0020	0.0025	0.0011
		SE	0.4554	0.5907	--	--	--	--
		RV	0.0739	0.0882	0.0578	0.0042	0.0070	0.0042
		Bias	0.2115	-0.1701	0.0220	-0.0179	-0.0245	0.0133
		RSE	0.1828	0.1766	0.3329	0.0931	0.1413	0.1267
48	24	**EV**	2.2489	2.8162	0.5268	0.4817	0.3451	0.2636
		MSE	0.1590	0.2002	0.0019	0.0016	0.0024	0.0009
		SE	0.3999	0.5091	--	--	--	--
		RV	0.0707	0.0711	0.0036	0.0034	0.0068	0.0035
		Bias	0.2489	-0.1838	0.0268	-0.0183	-0.0299	0.0136
		RSE	0.1773	0.1589	0.0825	0.0843	0.1406	0.1145

60	30	**EV**	2.2221	2.7531	0.5245	0.4766	0.3476	0.2675
		MSE	0.1175	0.1883	0.0015	0.0016	0.0018	0.0009
		SE	0.3532	0.4456	--	--	--	--
		RV	0.0529	0.0684	0.0028	0.0033	0.0052	0.0033
		Bias	0.2221	-0.2469	0.0245	-0.0234	-0.0274	0.0175
		RSE	0.1542	0.1576	0.0728	0.0832	0.1228	0.1103
72	36	**EV**	2.2340	2.7457	0.5261	0.4763	0.3458	0.2677
		MSE	0.1142	0.1655	0.0014	0.0014	0.0018	0.0008
		SE	0.3241	0.4059	--	--	--	--
		RV	0.0511	0.0603	0.0027	0.0029	0.0051	0.0029
		Bias	0.2340	-0.2544	0.0261	-0.0237	-0.0292	0.0177
		RSE	0.1513	0.1482	0.0716	0.0781	0.1218	0.1035
84	42	**EV**	2.2170	2.7208	0.5244	0.4742	0.3477	0.2693
		MSE	0.0968	0.1661	0.0012	0.0014	0.0015	0.0008
		SE	0.2975	0.3726	--	--	--	--
		RV	0.0437	0.0610	0.0023	0.0029	0.0044	0.0029
		Bias	0.2170	-0.2792	0.0244	-0.0258	-0.0273	0.0193
		RSE	0.1404	0.1498	0.0667	0.0788	0.1125	0.1035

Table 4.2: Maximum Likelihood Estimate of Parameters, Reliability and Hazard Rates and their Efficiency
Measures m = 3, α = 1.5, β_1 = 2, β_2 = 3, β_3 = 4,
$R(t)$ = (0.5,0.5,0.5), $h(t)$ = (0.375,0.250,0.1875)

u	G^*		$\widehat{\beta_1}$	$\widehat{\beta_2}$	$\widehat{\beta_3}$	$\hat{R}_1(t_1)$	$\hat{R}_2(t_2)$	$\hat{R}_3(t_3)$	$\widehat{h_1}(t_1)$	$\widehat{h_2}(t_2)$	$\widehat{h_3}(t_3)$
24	8	EV	2.3579	2.9827	3.7069	0.5357	0.4927	0.4745	0.3354	0.2554	0.2016
		MSE	0.3559	0.4316	0.8533	0.0038	0.0030	0.0040	0.0046	0.0016	0.0012
		SE	0.6638	0.8432	1.0573	--	--	--	--	--	--
		RV	0.1510	0.1447	0.2302	0.0070	0.0060	0.0084	0.0137	0.0063	0.0060
		Bias	0.3579	-0.0173	-0.2931	0.0357	-0.0073	-0.0255	-0.0396	0.0054	0.0141
		RSE	0.2530	0.2203	0.2492	0.1144	0.1103	0.1327	0.2024	0.1575	0.1724
36	12	EV	2.2134	3.0292	3.7877	0.5215	0.4982	0.4825	0.3510	0.2513	0.1973
		MSE	0.1901	0.3125	0.5169	0.0023	0.0021	0.0023	0.0028	0.0011	0.0007
		SE	0.5112	0.6989	0.8762	--	--	--	--	--	--
		RV	0.0859	0.1032	0.1365	0.0044	0.0042	0.0047	0.0080	0.0046	0.0036
		Bias	0.2134	0.0292	-0.2123	0.0215	-0.0018	-0.0175	-0.0240	0.0013	0.0098
		RSE	0.1970	0.1845	0.1898	0.0913	0.0916	0.0988	0.1511	0.1347	0.1345
48	16	EV	2.3233	2.9984	3.6469	0.5346	0.4966	0.4735	0.3364	0.2525	0.2023
		MSE	0.2183	0.2381	0.5118	0.0025	0.0016	0.0025	0.0031	0.0009	0.0008
		SE	0.4610	0.5990	0.7372	--	--	--	--	--	--
		RV	0.0940	0.0794	0.1403	0.0047	0.0033	0.0052	0.0093	0.0036	0.0038
		Bias	0.3233	-0.0016	-0.3531	0.0346	-0.0034	-0.0265	-0.0386	0.0025	0.0148
		RSE	0.2011	0.1627	0.1962	0.0938	0.0816	0.1049	0.1661	0.1194	0.1366
60	20	EV	2.3352	2.9872	3.6688	0.5364	0.4966	0.4757	0.3343	0.2526	0.2011
		MSE	0.2047	0.1726	0.4220	0.0024	0.0012	0.0020	0.0030	0.0007	0.0006
		SE	0.4147	0.5341	0.6642	--	--	--	--	--	--
		RV	0.0877	0.0578	0.1150	0.0044	0.0024	0.0042	0.0088	0.0026	0.0031
		Bias	0.3352	-0.0128	-0.3312	0.0364	-0.0034	-0.0243	-0.0407	0.0026	0.0136
		RSE	0.1938	0.1391	0.1771	0.0908	0.0694	0.0945	0.1625	0.1016	0.1244

72	24	**EV**	2.2735	3.0008	3.7056	0.5302	0.4981	0.4787	0.3412	0.2515	0.1994
		MSE	0.1464	0.1463	0.3434	0.0018	0.0010	0.0016	0.0022	0.0006	0.0005
		SE	0.3697	0.4894	0.6088	--	--	--	--	--	--
		RV	0.0644	0.0488	0.0927	0.0033	0.0020	0.0034	0.0065	0.0022	0.0025
		Bias	0.2735	0.0008	-0.2944	0.0302	-0.0020	-0.0213	-0.0338	0.0015	0.0119
		RSE	0.1683	0.1275	0.1581	0.0793	0.0637	0.0838	0.1376	0.0943	0.1122
84	28	**EV**	2.3473	3.0360	3.6501	0.5383	0.5011	0.4751	0.3321	0.2492	0.2014
		MSE	0.1867	0.1402	0.3464	0.0022	0.0009	0.0017	0.0028	0.0005	0.0005
		SE	0.3518	0.4587	0.5587	--	--	--	--	--	--
		RV	0.0795	0.0462	0.0949	0.0041	0.0019	0.0035	0.0083	0.0021	0.0026
		Bias	0.3473	0.0360	-0.3499	0.0383	0.0011	-0.0249	-0.0429	-0.0008	0.0139
		RSE	0.1841	0.1233	0.1612	0.0873	0.0608	0.0858	0.1580	0.0913	0.1129

Table 4.3: Maximum Likelihood Estimate of Parameters, Reliability and Hazard Rates and their Efficiency

Measures m = 2, α = 1.5, β_1 = 2, β_2 = 3, $R(t)$ = (0.5,0.5), $h(t)$ = (0.375,0.250)

u	G^*		$\hat{\alpha}$	$\widehat{\beta_1}$	$\widehat{\beta_2}$	$\hat{R}_1(t_1)$	$\hat{R}_2(t_2)$	$\hat{h}_1(t_1)$	$\hat{h}_2(t_2)$
12	6	**EV**	1.7436	2.0226	2.9095	0.4928	0.4807	0.4524	0.3251
		MSE	0.2837	0.3459	0.8842	0.0049	0.0060	0.0385	0.0279
		SE	0.4381	0.6965	1.0177	--	--	--	--
		RV	0.1627	0.1710	0.3039	0.0100	0.0125	0.0851	0.0857
		Bias	0.2436	0.0226	-0.0905	-0.0072	-0.0193	0.0774	0.0751
		RSE	0.3055	0.2908	0.3232	0.1426	0.1615	0.4336	0.5134
16	8	**EV**	1.6501	2.1126	2.9169	0.5060	0.4840	0.4105	0.3011
		MSE	0.1737	0.3002	0.6817	0.0036	0.0046	0.0218	0.0171
		SE	0.3597	0.6557	0.9205	--	--	--	--
		RV	0.1053	0.1421	0.2337	0.0072	0.0095	0.0530	0.0569
		Bias	0.1501	0.1126	-0.0831	0.0060	-0.0160	0.0355	0.0511
		RSE	0.2526	0.2593	0.2831	0.1189	0.1404	0.3594	0.4345
24	12	**EV**	1.5804	2.1489	2.8153	0.5136	0.4780	0.3825	0.2890
		MSE	0.0989	0.1822	0.4594	0.0023	0.0036	0.0110	0.0101
		SE	0.2827	0.5532	0.7356	--	--	--	--
		RV	0.0626	0.0848	0.1632	0.0044	0.0074	0.0289	0.0350
		Bias	0.0146	0.2369	-0.1935	0.0255	-0.0197	-0.0240	0.0208
		RSE	0.1990	0.1986	0.2407	0.0927	0.1248	0.2747	0.3482
36	18	**EV**	1.5347	2.2054	2.8201	0.5208	0.4806	0.3615	0.2752
		MSE	0.0514	0.1800	0.2977	0.0021	0.0024	0.0061	0.0052
		SE	0.2241	0.4677	0.6079	--	--	--	--
		RV	0.0335	0.0816	0.1056	0.0040	0.0049	0.0169	0.0189
		Bias	0.0347	0.2054	-0.1799	0.0208	-0.0194	-0.0135	0.0252
		RSE	0.1478	0.1924	0.1935	0.0880	0.1012	0.2162	0.2620

48	24	**EV**	1.5146	2.2369	2.8065	0.5255	0.4803	0.3510	0.2708
		MSE	0.0380	0.1494	0.2376	0.0018	0.0019	0.0046	0.0038
		SE	0.1918	0.4131	0.5276	--	--	--	--
		RV	0.0251	0.0668	0.0847	0.0034	0.0040	0.0130	0.0139
		Bias	0.0146	0.2369	-0.1935	0.0255	-0.0197	-0.0240	0.0208
		RSE	0.1286	0.1728	0.1737	0.0803	0.0916	0.1923	0.2263
60	30	**EV**	1.5016	2.2430	2.7920	0.5267	0.4798	0.3463	0.2676
		MSE	0.0309	0.1326	0.1886	0.0016	0.0015	0.0038	0.0027
		SE	0.1700	0.3721	0.4713	--	--	--	--
		RV	0.0206	0.0591	0.0675	0.0031	0.0032	0.0109	0.0101
		Bias	0.0016	0.2430	-0.2080	0.0267	-0.0202	-0.0287	0.0176
		RSE	0.1170	0.1623	0.1555	0.0764	0.0817	0.1778	0.1940
72	36	**EV**	1.4942	2.2334	2.7450	0.5260	0.4760	0.3454	0.2686
		MSE	0.0232	0.1139	0.1785	0.0014	0.0015	0.0032	0.0022
		SE	0.1546	0.3385	0.4232	--	--	--	--
		RV	0.0155	0.0510	0.0650	0.0027	0.0031	0.0092	0.0082
		Bias	-0.0058	0.2334	-0.2550	0.0260	-0.0240	-0.0296	0.0186
		RSE	0.1019	0.1511	0.1539	0.0715	0.0813	0.1635	0.1751
84	42	**EV**	1.4930	2.2574	2.7551	0.5288	0.4771	0.3423	0.2676
		MSE	0.0202	0.1237	0.1660	0.0015	0.0014	0.0032	0.0021
		SE	0.1431	0.3165	0.3931	--	--	--	--
		RV	0.0135	0.0548	0.0603	0.0029	0.0029	0.0094	0.0077
		Bias	-0.0070	0.2574	-0.2449	0.0288	-0.0229	-0.0327	0.0176
		RSE	0.0953	0.1558	0.1479	0.0737	0.0778	0.1657	0.1692

Table 4.4: Maximum Likelihood Estimate of Parameters, Reliability and Hazard Rates and their Efficiency
Measures m = 3, α = 1.5, β_1 = 2, β_2 = 3, β_3 = 4,
$R(t)$ = (0.5,0.5,0.5), $h(t)$ = (0.375,0.250,0.1875)

u	G^*		$\hat{\alpha}$	$\widehat{\beta_1}$	$\widehat{\beta_2}$	$\widehat{\beta_3}$	$\hat{R}_1(t_1)$	$\hat{R}_2(t_2)$	$\hat{R}_3(t_3)$	$\widehat{h_1}(t_1)$	$\widehat{h_2}(t_2)$	$\widehat{h_3}(t_3)$
24	08	**EV**	1.5636	2.3165	2.9795	3.6986	0.5304	0.4903	0.4709	0.3617	0.2773	0.2213
		MSE	0.0900	0.3573	0.5999	1.2497	0.0038	0.0040	0.0057	0.0121	0.0095	0.0074
		SE	0.0003	0.0008	0.0006	0.0006	--	--	--	--	--	--
		RV	0.0576	0.1542	0.2013	0.3379	0.0072	0.0083	0.0121	0.0335	0.0343	0.0336
		Bias	0.0636	0.3165	-0.0205	-0.3014	0.0304	-0.0097	-0.0291	-0.0133	0.0273	0.0363
		RSE	0.1919	0.2580	0.2600	0.3023	0.1166	0.1297	0.1604	0.3042	0.3518	0.3898
36	12	**EV**	1.5500	2.2045	3.0077	3.7994	0.5204	0.4952	0.4812	0.3670	0.2682	0.2107
		MSE	0.0605	0.1938	0.4109	0.7828	0.0022	0.0027	0.0033	0.0074	0.0059	0.0044
		SE	0.0002	0.0009	0.0012	0.0016	--	--	--	--	--	--
		RV	0.0390	0.0879	0.1366	0.2060	0.0043	0.0055	0.0070	0.0200	0.0221	0.0209
		Bias	0.0500	0.2045	0.0077	-0.2006	0.0204	-0.0049	-0.0188	-0.0080	0.0182	0.0257
		RSE	0.1586	0.1997	0.2131	0.2329	0.0911	0.1054	0.1203	0.2337	0.2872	0.3150
48	16	**EV**	1.5087	2.3689	3.0450	3.6965	0.5391	0.4998	0.4755	0.3360	0.2553	0.2065
		MSE	0.0378	0.2719	0.2950	0.6746	0.0029	0.0019	0.0030	0.0060	0.0035	0.0030
		SE	0.0002	0.0005	0.0009	0.0008	--	--	--	--	--	--
		RV	0.0250	0.1148	0.0969	0.1825	0.0055	0.0039	0.0064	0.0179	0.0136	0.0143
		Bias	0.0087	0.3689	0.0450	-0.3035	0.0391	-0.0002	-0.0245	-0.0390	0.0053	0.0215
		RSE	0.1288	0.2201	0.1784	0.2222	0.1007	0.0879	0.1157	0.2305	0.2306	0.2631
60	20	**EV**	1.4817	2.3804	3.0207	3.7269	0.5407	0.4983	0.4782	0.3289	0.2514	0.2006
		MSE	0.0303	0.2628	0.2541	0.5655	0.0029	0.0016	0.0026	0.0058	0.0026	0.0023
		SE	0.0002	0.0006	0.0007	0.0012	--	--	--	--	--	--
		RV	0.0204	0.1104	0.0841	0.1517	0.0054	0.0033	0.0054	0.0177	0.0105	0.0114
		Bias	-0.0183	0.3804	0.0207	-0.2732	0.0407	-0.0017	-0.0218	-0.0461	0.0014	0.0156
		RSE	0.1175	0.2154	0.1669	0.2018	0.0995	0.0815	0.1059	0.2319	0.2045	0.2380

72	24	**EV**	1.4818	2.3200	3.0686	3.7646	0.5348	0.5029	0.4816	0.3347	0.2476	0.1980
		MSE	0.0251	0.1963	0.2131	0.4409	0.0022	0.0013	0.0020	0.0047	0.0022	0.0017
		SE	0.0001	0.0005	0.0007	0.0010	--	--	--	--	--	--
		RV	0.0169	0.0846	0.0694	0.1171	0.0042	0.0027	0.0041	0.0140	0.0088	0.0087
		Bias	-0.0182	0.3200	0.0686	-0.2354	0.0348	0.0029	-0.0184	-0.0403	-0.0024	0.0130
		RSE	0.1069	0.1910	0.1504	0.1764	0.0883	0.0729	0.0921	0.2043	0.1885	0.2093
84	28	**EV**	1.4709	2.3763	3.0572	3.6847	0.5412	0.5025	0.4769	0.3250	0.2455	0.1984
		MSE	0.0205	0.2185	0.1698	0.4017	0.0025	0.0011	0.0019	0.0048	0.0018	0.0014
		SE	0.0002	0.0003	0.0004	0.0006	--	--	--	--	--	--
		RV	0.0139	0.0920	0.0555	0.1090	0.0047	0.0022	0.0040	0.0147	0.0072	0.0071
		Bias	-0.0291	0.3763	0.0572	-0.3153	0.0412	0.0025	-0.0231	-0.0500	-0.0045	0.0134
		RSE	0.0972	0.1967	0.1348	0.1720	0.0929	0.0664	0.0912	0.2123	0.1707	0.1893

Table 5.1: Likelihood Ratio Test for Testing $H_0 : \beta_1 = \beta_2 = \beta$ vs $H_1 : \beta_1 \neq \beta_2$ when $\alpha = 1.5, \beta_1 = 2, \beta_2 = 3$

u	G^*	β	$\widehat{\beta_1}$	$\widehat{\beta_2}$	LL_{H0}	LL_{H1}	χ^2	p-value
12	**6**	2.2060	1.8139	2.6251	0.3699	0.6329	0.5260	0.4683
24	**12**	2.7425	2.2361	3.3522	12.5759	13.2162	1.2805	0.2578
36	**18**	2.5671	2.1328	3.0856	35.3684	36.1586	1.5804	0.2087
48	**24**	2.4766	2.0968	2.9257	62.8023	63.6669	1.7292	0.1885
60	**30**	2.5674	2.1716	3.0594	82.9129	83.9998	2.1737	0.1404
72	**36**	2.6195	2.2404	3.0704	111.7497	112.8582	2.2171	0.1365
84	**42**	2.4682	2.1353	2.8637	151.6771	152.8280	2.3019	0.1292

Table 5.2: Likelihood Ratio Test for Testing $H_0 : \beta_1 = \beta_2 = \beta$ vs $H_1 : \beta_i \neq \beta_j$; $(i \neq j = 1, 2, 3)$ when $\alpha = 1.5, \beta_1 = 2, \beta_2 = 3$

u	G^*	β	$\widehat{\beta_1}$	$\widehat{\beta_2}$	$\widehat{\beta_3}$	LL_{H0}	LL_{H1}	χ^2	p-value
24	**8**	3.0603	2.6290	2.6392	4.0233	10.3662	11.1263	1.5200	0.4677
36	**12**	3.2628	2.4519	3.1456	4.2870	30.0355	31.5091	2.9472	0.2291
48	**16**	3.0863	2.2358	3.5430	3.5602	51.2363	52.9533	3.4340	0.1796
60	**20**	2.9991	2.1617	2.9642	4.0302	82.5253	85.4711	5.8915	0.0526
72	**24**	3.0498	2.1846	3.2678	3.7883	111.3079	114.2634	5.9111	0.0520
84	**24**	3.3133	2.2430	3.6661	4.1789	129.3180	133.7545	8.8730	0.0118

Table 6.1: Table of Total Time under Experiment, Duration of Experiment and Cost of Experiment Under ½ Censoring m = 2, α = 1.5, β_1 = 2, β_2 = 3

u	G	t_2	t_{max}	C_a^{II}
12	**12**	57.38	2.36	775.38
24	**24**	114.83	2.45	1428.82
36	**36**	173.80	2.51	2097.06
48	**48**	230.01	2.49	2737.04
60	**60**	289.14	2.50	3406.40
72	**72**	344.71	2.49	4039.98
84	**84**	402.78	2.51	4698.89

Table 6.2: Table of Total Time under Experiment, Duration of Experiment and Cost of Experiment Under 1/3 Censoring m = 3, α = 1.5, β_1 = 2, β_2 = 3, β_3 = 4

u	G	t_2	t_{max}	C_a^{II}
24	**24**	115.85	2.49	1439.42
27	**27**	119.07	2.24	1473.59
30	**30**	121.53	2.00	1500.30
33	**33**	123.55	1.84	1523.42
36	**36**	126.16	1.72	1552.82
39	**39**	128.82	1.59	1582.54
42	**42**	130.88	1.50	1606.85
45	**45**	133.94	1.42	1641.08
48	**48**	135.60	1.34	1661.38